SOLVABILITY OF THE PROBLEM OF PARALLEL-WALL CAVITATIONAL FLOW PAST A CURVED OBSTACLE

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Below it is shown how, in the simplest case, it is possible to use the Leray method [1] to prove the solvability of the problem of cavitational flow of a perfect incompressible fluid past a symmetric smooth arc in accordance with the Joukowski-Rosdhko scheme [2] in an infinite and a confined flow. The problem of uniqueness of the solution is not considered.

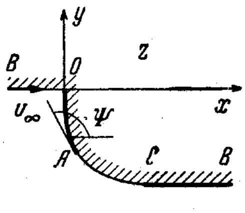

Fig. 1

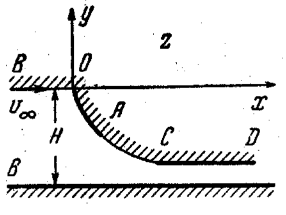

Fig. 2

1. The flow scheme will be clear from the drawings (Figs. 1, 2; the lower half of the flow is shown, OA represents the obstacle, AC the free jet). We shall derive the integral equations of the problems. As the auxiliary plane we shall take the plane $\zeta = \xi + i\eta$, in which the lower half of the flow corresponds to the quarter of the unit circle lying in the first quadrant; in this case the arc OA corresponds to the arc $\zeta = e^{is}$ $(0 \leqslant s \leqslant {}^1/_2\pi)$ and the free line to the segment $\zeta = \xi$ $(0 \leqslant \xi \leqslant 1)$.

The complex potential w is expressed in terms of ζ thus:

In the case of an infinite flow

$$w = a \frac{(1+\zeta^2)^2}{4\zeta^2 + b^2(1-\zeta^2)^2}\,. \tag{1}$$

Here $a > 0$ and b $(0 < b < 1)$ are unknown parameters ($b = 2\beta/(1+\beta^2)$), where $i\beta$ is the image in the plane ζ of the infinitely remote point B.

In the case of a confined (strip) flow

$$w = \frac{HV_\infty}{\pi} \ln \frac{(1-d^2)(4\zeta^2 + b^2(1-\zeta^2)^2)}{(1-b^2)(4\zeta^2 + d^2(1-\zeta^2)^2)} \qquad \left(b = \frac{2\beta}{1+\beta^2},\quad d = \frac{2\delta}{1+\delta^2}\right). \tag{2}$$

Here H is the half-width of the strip, V_∞ is the free stream velocity, b and d $(0 < d < b < 1)$ are unknown parameters, $i\beta$ and $i\delta$ are the images in the plane ζ of the infinitely remote points B and D.

By l we denote the arc abscissa of points on the obstacle OA (at the point O we have $l = 0$, at the point of separation of the jet $l = -l_0$); the angle between the tangent to the obstacle and the x axis is denoted by ψ. The function $\Psi\{l\}$ $(l_0 \leqslant l \leqslant 0)$ is known.

If we consider known the relation $l = l(s)$ $(0 \leqslant s \leqslant {}^1/_2\pi)$ associated with the conformal mapping of the quarter-circle onto the lower half of the flow, then for the function

$$\omega(\zeta) = i \ln dw/dz, \qquad \operatorname{Re}\omega(0) = (0), \tag{3}$$

we obtain by a well-known method (see, for example, [2]) the representation (V_0 is the velocity at the free line)

$$\omega(\zeta) = i \ln V_0 + i \ln \frac{1+i\zeta}{1-i\zeta} + \Omega(\zeta) \qquad (\Omega(\zeta) = \theta + iT). \tag{4}$$

The function $\Omega(\zeta)$ is continuous in the closed quarter-circle and satisfies the boundary conditions

$$\begin{gathered} T(\xi) = 0 \quad (0 \leqslant \xi \leqslant 1) \\ \theta(i\eta) = 0 \quad (0 \leqslant \eta \leqslant 1), \qquad \theta(e^{is}) = \Psi(l(s)\} - {}^1/_2\pi \quad (0 \leqslant s \leqslant {}^1/_2\pi)\,. \end{gathered} \tag{5}$$

Hence

$$\Omega(\zeta) = \frac{4\zeta(1-\zeta^2)}{\pi} \int_0^{{}^1/_2\pi} \frac{[\Psi\{l(s)\} - {}^1/_2\pi]\cos s}{(1+\zeta^2)^2 - 4\zeta^2\cos^2 s}\, ds\,. \tag{6}$$

Knowing $\omega(\zeta)$, with the aid of (3) and (1) or (2) it is easy to find $dz/d\zeta$. The boundary values of the modulus of this function for $\zeta = e^{is}$ $(0 \leqslant s \leqslant {}^1/_2\pi)$ are dl/ds.

Thus, we obtain an equation for the function $l(s)$

$$\frac{dl}{ds} = fe^{-T}\sin s \qquad (0 \leqslant s \leqslant {}^1/_2\pi), \qquad l\,({}^1/_2\pi) = 0\,. \tag{7}$$

Here T is the value of the imaginary part of function (6) for $\zeta = e^{is}$ $(0 \leqslant s \leqslant {}^1/_2\pi)$:

$$T = \frac{2}{\pi}\int\limits_0^{{}^1/_2\pi}\left[\Psi\{l(\sigma)\} - \frac{\pi}{2}\right]\frac{\sin s\cos\sigma}{\cos^2\sigma - \cos^2 s}\,d\sigma\,. \tag{8}$$

In the case of an infinite flow

$$f = f(s;\,a,\,b) = \frac{2a\,(1-b^2)}{V_0}\,\frac{1+\sin s}{(1-b^2\sin^2 s)^2}\,. \tag{9}$$

In the case of strip flow

$$f = f(s,\,b,\,d) = \frac{2HV_\infty}{\pi V_0}(b^2 - d^2)\,\frac{1+\sin s}{(1-b^2\sin^2 s)(1-d^2\sin^2 s)}\,. \tag{10}$$

In order to determine the unknown parameters (a, b or b, d) we make use of two additional conditions:

The free stream velocity

$$\omega\,(i\beta) = i\ln V_\infty \tag{11}$$

is known.

The length of the arc

$$l\,(0) = -\,l_0 \tag{12}$$

is given.

2. We shall assume that the arc is such that

(a) $\Psi\{0\} = {}^1/_2\,\pi, \qquad |\,\Psi\{l\} - {}^1/_2\,\pi\,| \leqslant {}^1/_2\,p\pi \qquad (p < 1),$

(b) The function $\psi\,\{l\}$ satisfies a Hölder condition.

By some means we extend $\psi\,\{l\}$ to the entire axis $-\infty < l < +\infty$, so that conditions (a) and (b) are satisfied.

In reference [3] the following existence theorem for solutions of equations of type (7) was proven.

Theorem 1. In Eq. (7) let the function T be given by Eq. (8), let the function $\Psi\{l\}$ $(-\infty < l < +\infty)$ satisfy conditions (a) and (b), and let the function $f = f(s;\,\lambda,\,\mu)$ $(0 \leqslant s \leqslant {}^1/_2\pi,\ \lambda_1 \leqslant \lambda \leqslant \lambda_2,\ \mu_1 \leqslant \mu \leqslant \mu_2)$ be continuous. The parameters[1] λ, μ are found together with $l\,(x)$, for which two additional conditions relating $\Psi\{l\,(s)\}$, λ and μ are specified. Let these conditions be such that for an arbitrary function $l\,(s)$ satisfying a Hölder condition they define λ and μ on the segments $[\lambda_1, \lambda_2]$, $[\mu_1, \mu_2]$ uniquely, the values of the parameters thus found depending continuously on $l\,(s)$: for small variations in max $|\,l\,(s)\,|$ and the Hölder constant of $l\,(s)$ the values of the parameters change only slightly. Then Eq. (7) has a solution.

The proof of this theorem, which essentially reproduces the reasoning of Leray [1], consists briefly in the following. The equation is examined in the space E_ν of functions vanishing at $s = \pi/2$ and satisfying the Hölder condition[2]

$$\sup_{0\leqslant s_1, s_2\leqslant {}^1/_2\pi}\frac{|\,l\,(s_1) - l\,(s_2)\,|}{|\cos s_1 - \cos s_2|^\nu} = C_l < \infty$$

with the norm $\|\,l\,(s)\,\| = \max|\,l\,(s)\,| + C_l$ introduced in the usual way. The exponent ν is chosen so that the condition

$$0 < \nu < 1 - p \tag{13}$$

is satisfied.

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[1] There may, of course, be any number of them. The number of additional conditions coincides with the number of parameters.

[2] Equations of type (7) have been studied in other spaces also [4, 5], but for our purposes it is most convenient to take the space E_ν considered by Leray.

Equation (7) is treated as the problem of the fixed point of an operator A that acts in accordance with the formula

$$L(s) = Al(s) = -\int_{s}^{1/2\pi} f(s, \lambda, \mu) e^{-T} \sin s ds, \tag{14}$$

Where T is calculated from (8), and λ and μ are found from $l(s)$ using the additional conditions.

We also introduce the operator $X[l(s), t]$, obtained from A by replacing the function $\Psi\{l\}$ everywhere with the function

$$t\Psi\{l\} + {}^1/_2(1-t)\pi. \tag{15}$$

At t = 1 this operator goes over into A, and at t = 0 into an operator that carries the entire space E_ν into a single function.

By virtue of the Leray-Schauder fixed point principle [6], equation (7) has a solution if the operator X is completely continuous on E_ν in the interval $0 \leqslant t \leqslant 1$, and solutions of the equation $l(s) = X\{l(s); t\}$ are totally bounded with respect to the norm of E_ν.

The complete continuity of X is easily verified. The boundedness of the set of solutions is proved as follows.

Obviously, by virtue of the normalization $l({}^1/_2\pi) = 0$, it is sufficient to prove the inequality

$$|l(s_1) - l(s_2)| \leqslant C|\cos s_1 - \cos s_2|^\nu \qquad (0 \leqslant s_1, s_2 \leqslant {}^1/_2\pi),$$

where C is the same for all solutions. According to the Hölder inequality

$$|l(s_1) - l(s_2| \leqslant \left|\int_{s_1}^{s_2} \left|\frac{dl}{d\cos s}\right|^{1/(1-\nu)} ds\right|^{1-\nu} |\cos s_1 - \cos s_2|^\nu .$$

Using (7), we find that it is sufficient to show the boundedness of the integrals

$$\int_0^{1/2\pi} \exp\frac{-T}{1-\nu} ds, \tag{16}$$

where T is calculated from (8), in which $\Psi\{l\}$ is replaced with function (15).

By the principle of symmetry the function $\Omega(\zeta)$ can be extended to the entire unit circle. In accordance with the law of the mean

$$\int_0^{2\pi} \exp\frac{-T}{1-\nu} \cos\frac{\theta}{1-\nu} ds = 2\pi .$$

Hence, by virtue of (5),

$$\int_0^{\pi} \exp\frac{-T}{1-\nu} \cos\frac{\Psi\{l(s)\} - {}^1/_2\pi}{1-\nu} ds = \pi .$$

Taking into account condition (a), imposed on $\Psi\{l\}$, and the choice of the number ν (13), we obtain

$$\left|\frac{\Psi - {}^1/_2\pi}{1-\nu}\right| \leqslant \frac{p}{1-\nu}\frac{\pi}{2} < \frac{\pi}{2}.$$

Hence it is easy to obtain an estimate for the integral (16) depending only on p and ν. It remains only to point out that if $\Psi\{l\}$ satisfies the condition (a), then this condition is also satisfied by the function (15) for any t.

3. Theorem 1 makes it possible to obtain the existence theorem quite simply for different problems of the theory of jets and for more general problems where the velocity at unknown parts of the boundary of the flow region is variable [3].

We shall test the fulfillment of the conditions of the theorem for the case of an infinite flow between parallel walls.

In this case the additional conditions (11) and (12) can be written in the following form:

Condition (11), with account for (4) and (6), becomes

$$\frac{2}{\pi}\int_0^{1/2\pi}[\pi-\Psi\{l(s)\}]\frac{b\cos s}{1-b^2\sin^2 s}\,ds=\ln\frac{V_0}{V_\infty} \tag{17}$$

Condition (12), by virtue of (7) and (9), becomes

$$\frac{2a(1-b^2)}{V_0}\int_0^{1/2\pi}\frac{(1+\sin s)\sin s}{(1-b^2\sin^2 s)^2}\,e^{-T}\,ds=l_0\,. \tag{18}$$

For an arbitrary function $l(s)$ satisfying a Hölder condition these two equations uniquely define the values of the parameters a and b. Indeed, from condition (a) imposed on the function $\psi\{l\}$, it is easy to establish that with variation of b from 0 to 1 the left side of (17) increases monotonically from zero to infinity. Therefore, for any fixed V_0, V_∞ ($V_0 \geq V_\infty$) and $l(s)$, Eq. (17) has an equation in b as a unique solution. After b has been found, a is found from (18).

It is easy to verify the continuous dependence of a and b on $l(s)$.

We shall now show that there are estimates $b \leq b_0 < 1$, $a \leq a_0$ for the values of a and b found from (17) and (18) that do not depend on $l(s)$. From condition (a) it follows that the left side of (17) for any b is not less than $1/2(1-p)\ln(1+b)/(1-b)$, so that as b_0 we can take the root of the equation

$$\frac{1-p}{2}\ln\frac{1+b_0}{1-b_0}=\ln\frac{V_0}{V_\infty}\,.$$

The second estimate may be obtained as follows. It is easy to verify that

$$\int_0^{1/2\pi}T\sin s\,ds=\int_0^{1/2\pi}\left[\Psi\{l(s)\}-\frac{\pi}{2}\right]\cos s\,ds,$$

so that

$$\left|\int_0^{1/2\pi}T\sin s\,ds\right|\leqslant p\frac{\pi}{2}.$$

We have the inequality (a consequence of the fact that the graph of the function $y = e^x$ is turned convex side down)

$$\int_0^{1/2\pi}e^{-T}\sin s\,ds>\exp\left(-\int_0^{1/2\pi}T\sin s\,ds\right).$$

From (18) we now get the required estimate; in this case

$$a_0=\frac{l_0V_0\exp(1/2\,p\pi)}{2(1-b_0^2)}\,.$$

It remains to point out that when $0\leqslant s\leqslant 1/2\pi$, $0\leqslant b\leqslant b_0$, $0\leqslant a\leqslant a_0$ the function (9) is continuous. The following result is obtained.

Theorem 2. **The problem of Joukowskii-Roshko flow past a symmetrical obstacle satisfying conditions $(a)_1(b)_1$ is solvable for any positive cavitation number.**

In the case of strip flow the additional conditions (11) and (12), after expansion, have the form (17) and

$$\frac{2HV_\infty}{\pi V_0}(b^2-d^2)\int_0^{1/2\pi}\frac{(1+\sin s)\sin s}{(1-b^2\sin^2 s)(1-d^2\sin^2 s)}\,e^{-T}\,ds=l_0\,.$$

If $l(s)$ is given, then b is found uniquely from (17) (in this case $b \leq b_0$, where b_0 has the same value as before). For d varying from 0 to b, the left side of (19) decreases monotonically from a certain value (depending on $l(s)$) to zero. Therefore Eq. (19), considered as an equation for d, has not more than one solution. But this may be unsolvable. In this case we agree to take d = 0. It is easy to verify that the parameter d thus defined depends continuously on $l(s)$. For $0 \leq s \leq \pi/2$, $0 \leq d \leq b \leq b_0$ the function (10) is continuous.

Note that when d = 0 Eq. (7) goes over into the equation for classical Kirchhoff flow. Therefore application of Theorem 1 gives the following theorem.

Theorem 3. Consider symmetrical strip flow when the arc satisfies conditions (a), (b). Then for any positive cavitation number at least one of the following problems is solvable: 1) Joukowski-Roshko flow with separation at the ends of the arc; 2) Kirchhoff flow with separation at the ends or at interior points of the arc.

The question of satisfaction of the Brillouin conditions (e.g., [2]) is not considered here.

We assume that the obstacle has the following natural property: for Kirchhoff flow with separation at interior points the larger cavitation number corresponds to later separation. Then Theorem 3 assumes the following form:

Theorem 3'. Consider symmetrical strip flow when the arc satisfies conditions (a), (b). Let Q_0 be the cavitation number corresponding to Kirchhoff flow (with separation at the ends of the arc). Then for this arc the problem of Joukowski-Roshko flow (with separation at the ends of the arc) has a solution for any cavitation number greater than Q_0.

Note that exactly the same result holds for Ryabushinskii flow [3].

It is easy to extend the theorems obtained to the case of flow past a symmetrical wedge with curved cheeks (see [7]).

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